TRINITY COLLEGE

Mathematical Methods Units 3.4
Test 5 2018

Section 2 Calculator Assumed

Continuous Random Variables

STUDENT'S NAME ______________________________

DATE: Thursday 9 August **TIME:** 50 minutes **MARKS**: 52

INSTRUCTIONS:

Standard Items: Pens, pencils, drawing templates, eraser

Special Items: Three calculators, notes on one side of a single A4 page (these notes to be handed in with this assessment)

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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1. (3 marks)

The probability density function of the random variable Y is

$$f(y)=\begin{cases}\dfrac{6}{25}-ky & 2\le y\le 12\\[2ex] 0 & \textit{elsewhere}\end{cases}$$

Determine the value of k.

$$\int_2^{12} \frac{6}{25} - ky = 1$$

$$\left[\frac{6y}{25} - \frac{ky^2}{2}\right]_2^{12} = 1$$

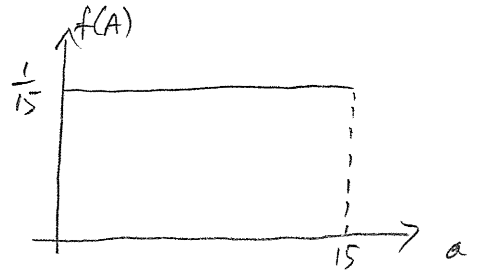
$$\left(\frac{72}{25} - \frac{72k}{2}\right) - \left(\frac{12}{25} - 2k\right) = 1$$

$$k = \frac{1}{50}$$

2. (9 marks)

Peter gets picked up after school by his mother each day. Peter's mother's arrival time is uniformly distributed between 3.20pm and 3.35pm.
Let A be the number of minutes after 3.20pm that Peter's mother arrives at his school to pick him up.

(a) Sketch the probability distribution function for A. [1]



(b) Write down the probability distribution function for A. [1]

$$f(A) = \begin{cases} \frac{1}{15} & 0 \le a \le 15 \\ 0 & \text{ELSEWHERE} \end{cases}$$

(c) Determine the probability that Peter's mother picks him up at 3.30 pm on both Monday and Tuesday next week. [1]

0

(d) Peter has a dentist appointment after school tomorrow at 3.45 pm. It takes 14 minutes to get to the dentist from school. Determine the probability that Peter will arrive at the dentist on time. [2]

MUST BE PICKED UP BY 3.31

$\therefore \frac{11}{15}$ 0.73

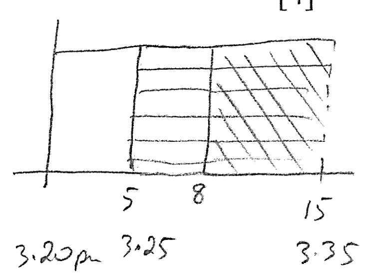
(e) Peter tells his mother he will be ready to be picked up at 3.25 pm for every day in the next school week. What is the probability he will have to wait at least 3 minutes on at least 2 occasions. [4]

$P(A \geq 8 \mid A \geq 5) = \frac{7}{10}$

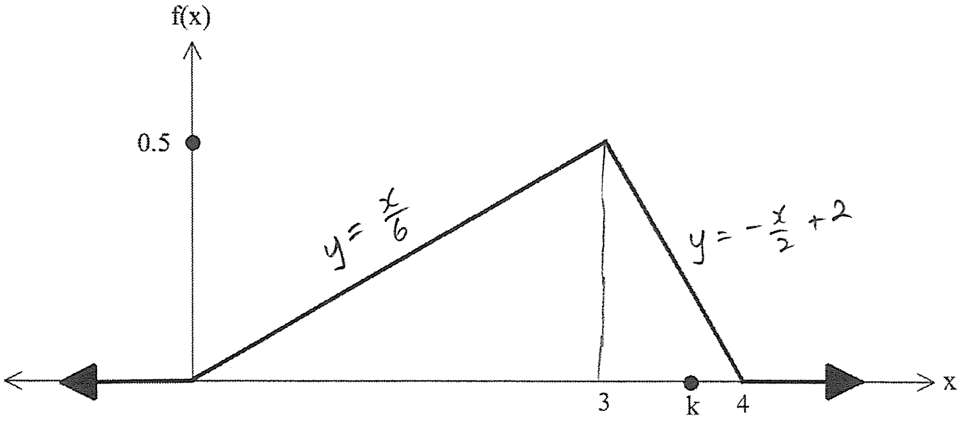
$Y \sim b(5, 0.7)$

$P(Y \geq 2) = b(2, 5, 5, 0.7)$

$= 0.9692$



3. (8 marks)



The diagram shows the probability density function of the continuous random variable X. Given P(X < k) = 0.96 and f(3) = 0.5,

(a) show k = 3.6 [3]

$(3, 0.5) \quad (4, 0) \quad y = -\frac{x}{2} + 2$

$$\int_k^4 -\frac{x}{2} + 2 \, dx = 0.04$$

$$\left[\frac{-x^2}{4} + 2x \right]_k^4 = 0.04$$

$$-4 + 8 - \left(-\frac{k^2}{4} + 2k\right) = 0.04$$

$$k = \frac{18}{5}, \cancel{\frac{22}{5}}$$

$$= 3.6$$

(b) determine P(X > 2) [2]

$y = \frac{x}{6}$

$x = 2 \quad y = \frac{1}{3}$

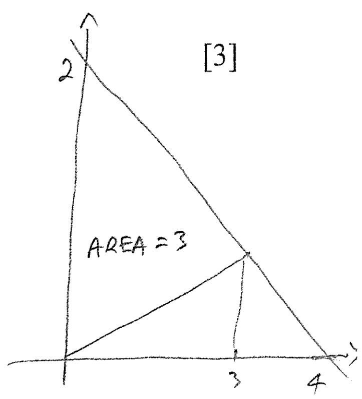
$1 - P(X < 2)$

$= 1 - \frac{1}{2} \times 2 \times \frac{1}{3}$

$= \frac{2}{3}$

(c) give the cumulative density function of X. [3]

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x^2}{12} & 0 \leq x \leq 3 \\ -\frac{x^2}{4} + 2x - 3 & 3 < x \leq 4 \\ 1 & x > 4 \end{cases}$$

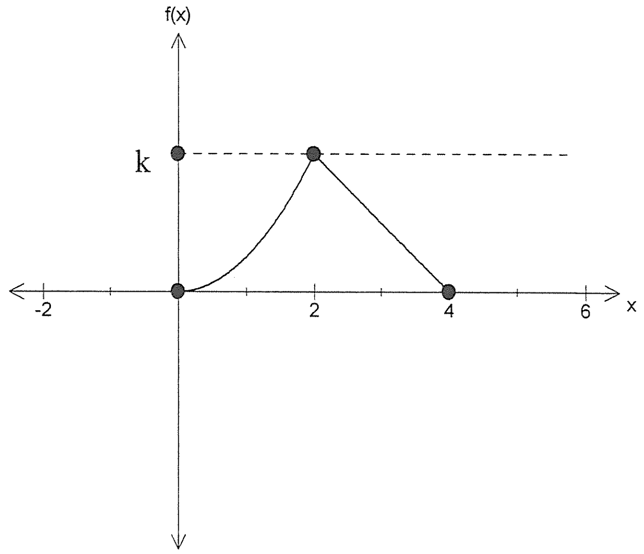


4. (7 marks)

The function $f(x)$ shown below is potentially the probability density function for a continuous random variable.

(a) Determine the value(s) of k which will make it a valid p.d.f. [3]

$$f(x) = \begin{cases} \frac{k}{4}x^2 & ; 0 \le x \le 2 \\ \frac{-k}{2}x + 2k & ; 2 \le x \le 4 \\ 0 & ; otherwise \end{cases}$$



$$\int_0^2 \frac{kx^2}{4}\,dx + \int_2^4 -\frac{kx}{2} + 2k\,dx = 1$$

$$\left[\frac{kx^3}{12}\right]_0^2 + \left[-\frac{kx^2}{4} + 2kx\right]_2^4 = 1$$

$$\frac{8k}{12} + (-4k + 8k) - (-k + 4k) = 1$$

$$k = \frac{3}{5}$$

(b) Calculate the median of this distribution [4]

$$\int_0^2 \frac{3}{5} \times \frac{x^2}{4}\,dx = \frac{2}{5}$$

$$\frac{1}{2} - \frac{2}{5} = \frac{1}{10}$$

$$\int_2^m -\frac{3}{5} \times \frac{x}{2} + 2 \times \frac{3}{5}\,dx = \frac{1}{10}$$

$$\left[\frac{-3x^2}{20} + \frac{6x}{5}\right]_2^m = \frac{1}{10}$$

$$\left(\frac{-3m^2}{20} + \frac{6m}{5}\right) - \left(\frac{-12}{20} + \frac{12}{5}\right) = \frac{1}{10}$$

$$m = 2.174, \cancel{5.826}$$

5. (8 marks)

A continuous random variable, X, has the probability density function

$$g(x) = \begin{cases} p + qx^2 & \text{for } 0 \le x \le 2 \\ 0 & \textit{elsewhere} \end{cases}$$

(a) If it is found that $P(X \le 1) = 0.2$, determine the values of the variables p and q. [4]

$\int_0^1 p + qx^2 \, dx = 0.2$

$\left[px + \frac{qx^3}{3}\right]_0^1 = 0.2$

$p + \frac{q}{3} = 0.2$

$\int_0^2 p + qx^2 \, dx = 1$

$\left[px + \frac{qx^3}{3}\right]_0^2 = 1$

$2p + \frac{8q}{3} = 1$

$p = \frac{1}{10}$

$q = \frac{3}{10}$

(b) Hence determine E(X), the expected value of X [2]

$\int_0^2 x(0.1 + 0.3x^2) \, dx$

$= 1.4$

(c) Calculate VAR(X), the variance of X. [2]

$\int_0^2 (0.1 + 0.3x^2)(x - 1.4)^2 \, dx$

$= 0.227$

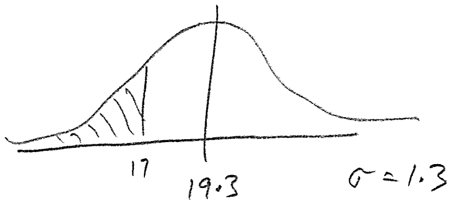
$\frac{17}{75}$

6. (13 marks)

Milan rides his motorbike to his part time job and home on those days when he works. The time for his ride each way is X minutes. X is a normally distributed random variable with a mean of 19.3 minutes and a standard deviation of 1.3 minutes.

(a) Determine $P(X < 17)$ [1]

$= 0.0384$



(b) Determine k if $P(\mu - k \le X \le \mu + k) = 0.3456$ where μ is the mean time. [3]

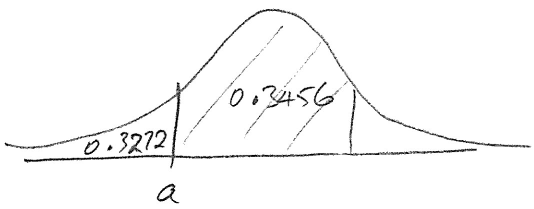
$P(X < a) = 0.3272$

$a = 18.7$

$k = \mu - a$

$= 19.3 - 18.7$

$= 0.6$



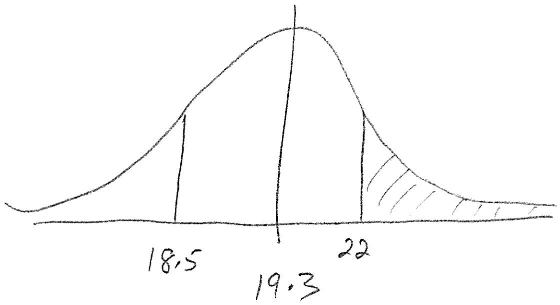
(c) One day when Milan sets off to work he knows he has 22 minutes to get there without being late. When stopped at a red traffic light he checks his watch and notices he has been travelling for 18.5 minutes. What is the probability he will be late? [3]

$P(X > 22 \mid X > 18.5)$

$= \dfrac{P(X > 22)}{P(X > 18.5)}$

$= \dfrac{0.0189}{0.7308}$

$= 0.0259$



(d) Calculate the probability that Milan will take between 18 and 20 minutes to get to work and also the same time interval to return home after work. [2]

$\left(P(18 < X < 20)\right)^2$

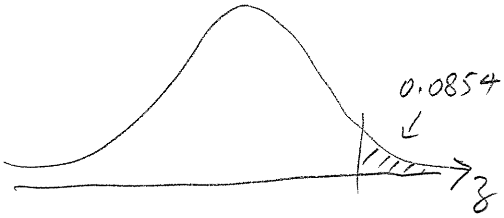
$= 0.5462^2$

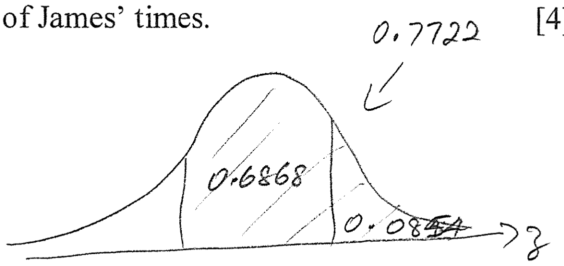
$= 0.2983$

(e) James has been monitoring his travelling times to his part time job and provided the following statistics of his normally distributed times.

$P(X > 27) = 0.0854$ and $P(21 < X < 27) = 0.6868$

Determine the mean and standard deviation of James' times. [4]





$z = 1.369$ $z = -0.746$

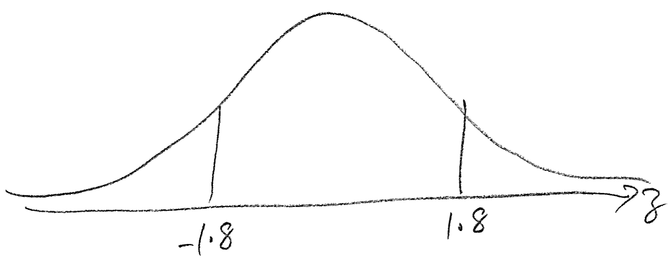
$$z = \frac{x-\mu}{\sigma}$$

$$1.369 = \frac{27-\mu}{\sigma} \qquad -0.746 = \frac{21-\mu}{\sigma}$$

$$\mu = 23.12$$
$$\sigma = 2.84$$

7. (4 marks)

The Exquisite Gem Company collect river pebbles which are then polished and sold. It has been found that the weight of these polished pebbles is normally distributed, where the standard deviation is 15% of the value of the mean. The Company cannot use pebbles that are more than 1.8 standard deviations from the mean as they are either too large or too small. In fact the largest pebble used weighs 94.6 grams. Calculate the weight of the smallest pebble used.



$\mu = \mu$

$\sigma = 0.15\mu$

$$z = \frac{x - \mu}{\sigma}$$

$$1.8 = \frac{94.6 - \mu}{0.15\mu}$$

$\mu = 74.49$

$\sigma = 11.17$

$$-1.8 = \frac{x - 74.49}{11.17}$$

$x = 54.38$